AIAA JOURNAL
Vol. 42, No. 9, September 2004

Technical Notes

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Convergence of Boobnov–Galerkin Method Exemplified

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I. Introduction

BEGINNING as a seed of an idea by Boobnov, in 1913, and a paper by Galerkin,[1] a mere two years later, numerous investigators in various fields of engineering sciences had adopted the technique the two researchers had derived, owing to its unusual potential.[2,3] Boobnov–Galerkin's method has been widely used in several fields of mechanical sciences as an approximate method to solve boundary value problems. Yet, despite its wide use, there are a very limited number of cases in which the convergence was directly demonstrated.

The Boobnov–Galerkin method was proven to converge[4] to the exact solutions for a large class of mechanical problems.[5–9] A pertinent description of the method and actual applications are reported in Ref. 1. Moreover, the equivalence of the Boobnov–Galerkin and Raylegh–Ritz methods was shown by Singer[10] and several other investigators for certain conditions. Elishakoff and Lee[11] demonstrated that, for uniform beams, simply supported at both ends, the Boobnov–Galerkin's method and the Fourier series method lead to identical solutions. Also note that Galerkin[1] considered bending of uniform beams clamped at both ends using the set of functions

$$P_j(x) = 1 - (-1)^j \cos(2j\pi x/L) \tag{1}$$

where $P_j(x)$ are comparison functions, L the length of the beam, j the serial number of the comparison function, and x the axial coordinate measured from the midspan of the beam. Galerkin summed up the series resulting from his method and showed that the results coincided with the exact solution, obtainable by direct integration.

In this Note, Boobnov–Galerkin's method is proved to converge to an exact solution for an applied mechanics problem. We address in detail the interrelation of Boobnov–Galerkin method and the exact solution in the beam deflection problems. Namely, we show the coincidence of these two methods for clamped–clamped boundary conditions, using an alternative set of functions proposed by Filonenko-Borodich.[12]

Received 25 February 2003; accepted for publication 13 March 2004.
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We consider a particular case of transverse concentrated load. The series is summed up and shown to coincide with well-known closed-form solutions.

II. Clamped–Clamped Beam Under Concentrated Load

Let us consider a clamped–clamped uniform beam under static transverse load $q(x)$. The differential equation of the transverse deflection $w(x)$ of the beam reads

$$EI\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x) \tag{2}$$

with E and I, respectively, the modulus of elasticity and the moment of inertia of the beam cross section with respect to the neutral axis. The boundary conditions associated with the transverse displacement $w(x)$ read

$$w(x) = \frac{\mathrm{d}w(x)}{\mathrm{d}x}, \qquad x = 0, \qquad x = L \tag{3}$$

L being the length of the beam.

The Boobnov–Galerkin solution of Eq. (2) is achieved when an approximate solution $\tilde{w}_n(x)$ is chosen in the series form

$$\tilde{w}_n(x) = \sum_{j=0}^{n} A_j P_j(x) \tag{4}$$

Herein, the comparison function $P_j(x)$ is represented in the following form:

$$P_{2j}(x) = \cos(2j\pi x/L) - \cos[2(j+1)\pi x/L] \tag{5}$$

$$P_{2j+1}(x) = \cos[(2j+1)\pi x/L] - \cos[(2j+3)\pi x/L] \tag{6}$$

where a distinction has been made between the symmetric Eq. (5) and asymmetric Eq. (6) functions $P_j(x)$ with respect to the mid span of the beam. The set of functions $P_j(x)$, $j = 1, 2, \ldots, \mathrm{int}[n/2]$ constitutes a class of complete functions in the range $[0, L]$ (see Stepanov[13]) that satisfy the boundary conditions in Eq. (3); int[•] indicates integer part. (This set was apparently first introduced by Filonenko-Borodich.[12]) Moreover, functions $P_j(x)$, $j = 1, 2, \ldots$, are quasi-orthogonal; that is, the following conditions hold:

$$\int_0^L \frac{\mathrm{d}^4 P_j(x)}{\mathrm{d}x^4} P_k(x)\,\mathrm{d}x = -\frac{\pi^4}{2L^3}\begin{cases} j^4 & \text{if} \quad j-k=2 \\ (j+2)^4 & \text{if} \quad k-j=2 \end{cases}$$

$$\int_0^L \frac{\mathrm{d}^4 P_j(x)}{\mathrm{d}x^4} P_k(x)\,\mathrm{d}x = \begin{cases} \dfrac{\pi^4[j^4+(j+2)^4]}{2L^3} & \text{if} \quad j=k \\ 0 & \text{if} \quad j \neq k \end{cases} \tag{7}$$

Let us assume that the load along the beam axis is represented by a concentrated load applied at distance a from the left end of the beam. Load distribution $q(x)$ is represented as

$$q(x) = Q\delta(x-a), \qquad 0 \le a \le L \tag{8}$$

where $\delta(x)$ is the Dirac delta function, satisfying the relationships

$$\int_0^L \delta(x - x_0)\,\mathrm{d}x = 1 \tag{9a}$$

$$\int_0^L \delta(x - x_0) f(x)\,\mathrm{d}x = f(x_0) \tag{9b}$$

Transverse displacement of the beam cross section, specifically for $a = L/2$, that is, when a concentrated load is applied at the midspan, is given by the known expression

$$w(L/2) = PL^3/192\,EI \tag{10}$$

We use the approximation in Eq. (4) to obtain the Boobnov–Galerkin's solution. In these circumstances, the error $\varepsilon_n(x)$ reads

$$\varepsilon_n(x) = EI \sum_{j=1}^{n} A_j \frac{\mathrm{d}^4 P_j(x)}{\mathrm{d}x^4} - P\delta(x - L/2) \tag{11}$$

The requirement that the product $[\varepsilon_n(x), P_j(x)] = 0$, $j = 1, 2, \ldots, n$ vanishes yields a system of n equations in the n unknowns A_j, $j = 1, 2, \ldots, n$, that may be recast for odd and even comparison functions $P_{2j+1}(x)$ and $P_{2j}(x)$, respectively, as

$$\begin{aligned} &A_0 - A_2 = q_0 L^4/8EI \equiv K \\ &2^4(A_2 - A_4) - (A_0 - A_2) = 0 \\ &\ldots \\ &(j+1)^4(A_{2j} - A_{2j+2}) - j^4(A_{2j-2} - A_{2j}) = 0 \\ &\ldots \end{aligned} \tag{12}$$

$$\begin{aligned} &A_1 + 3^4(A_1 - A_3) = 0 \\ &3^4 A_3 + 5^4(A_3 - A_5) = 0 \\ &\ldots \\ &(j+1)^4 A_{2j+1} + (j+3)^4(A_{2j+1} - A_{2j+3}) = 0 \\ &\ldots \end{aligned} \tag{13}$$

where Eq. (7) has been taken into account and integrals that involve terms $q(x)$ have been evaluated, yielding

$$\int_0^L q(x) P_j(x)\,\mathrm{d}x = Q P_j\left(\frac{L}{2}\right) \tag{14}$$

Solution of the algebraic system in Eq. (13) is trivial, yielding, for an odd-numbered constant, the solution

$$A_1 = A_3 = \cdots = A_{2j-1} = \cdots = 0 \tag{15}$$

The even-numbered constants are given by the following expressions:

$$A_0 = \frac{FL^3}{EI\pi^4} \sum_{j=0,2,\ldots}^{n} \frac{1}{(j+1)^4}$$

$$A_{2j} - A_{2(j+1)} = \frac{FL^3}{EI\pi^4}\frac{1}{(j+1)^4}, \qquad j = 0, 2, \ldots, n \tag{16}$$

As a result, the transverse displacement $\tilde{w}_n(x)$ becomes

$$\tilde{w}_n(x) = \sum_{j=1}^{n} A_j P_j(x) = \frac{FL^3}{4\pi^4 EL}\left\{\sum_{j=0,2,4,\ldots}^{n} \frac{1}{(j+1)^4} - \sum_{j=0,2,4,\ldots}^{n} \frac{\cos[2(j+1)\pi x/L]}{(j+1)^4}\right\} \tag{17}$$

For the applied external load at $x = L/2$, we obtain

$$\begin{aligned} \tilde{w}_n\left(\frac{L}{2}\right) &= \frac{FL^3}{4\pi^4 EI}\left(\sum_{j=0,2,4,\ldots}^{n} \frac{1}{(j+1)^4} + \sum_{j=0,2,4,\ldots}^{n} \frac{(-1)^j}{(j+1)^4}\right) \\ &= \frac{FL^3}{2\pi^4 EI}\left(\sum_{j=0,2,4,\ldots}^{n} \frac{1}{(j+1)^4}\right) \end{aligned} \tag{18}$$

When the number of retained terms in Eq. (18), $n \to \infty$, Eq. (18) takes the form[14]

$$\tilde{w}_\infty\left(\frac{L}{2}\right) = \frac{FL^3}{2\pi^4 EI} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^4} = \frac{FL^3}{192EI} \tag{19}$$

which coincides with the well-known expression obtained by direct integration of Eq. (2) with load $q(x)$ expressed in Eq. (22).

Bending moment $M(x)$, along the beam axis, can be evaluated by means of Eq. (19), when the familiar relation is borne in mind:

$$\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = -\frac{M(x)}{EI} \tag{20}$$

which, being applied to Eq. (19), yields

$$\tilde{M}_n(x) = -\frac{FL}{\pi^2} \sum_{j=0,2,4,\ldots}^{n} \frac{\cos[2(j+1)\pi x/L]}{(j+1)^2} \tag{21}$$

Equation (21) yields, when $n \to \infty$,

$$\begin{aligned} \tilde{M}_\infty(x) &= -\frac{FL}{\pi^2} \sum_{j=1}^{\infty} \frac{\cos[(2j-1)\pi x/L]}{(2j-1)^2} \\ &= \frac{FL}{\pi^2}\left(\frac{\pi^2}{6} - \frac{\pi^2 x}{L} + \frac{\pi^2 x^2}{L^2}\right) \end{aligned} \tag{22}$$

Evaluation $\tilde{M}_\infty(0)$ or $\tilde{M}_\infty(L)$ results in the following expressions:

$$\tilde{M}_\infty(0) = \tilde{M}_\infty(L) = -FL/6 \tag{23}$$

coalescing with the formula for the bending moment, evaluated by direct solution of Eq. (2).

III. Summary

For conservative systems, it has been shown that the Boobnov–Galerkin's method and Rayleigh–Ritz method coincide. Hence, the preceding derivation also proves coincidence of the Rayleigh–Ritz and the closed-form solutions. Although the convergence of the Boobnov–Galerkin method has been proven for more general classes of problems,[4–9] the actual demonstration of the coincidence with universal known closed-form solutions appears worthwhile and instructive.

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Analysis of Time-Variant Aeroelastic Systems Using Neural Network

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Introduction

THE use of smart materials[1] in wings allows active control of the deflections of the wing structure. By using adaptive material in aircraft wings, it is possible to introduce the property of time-varying stiffness. Another approach to a variable stiffness wing is through the concept of a variable stiffness spar,[2] which allows for the variation of the torsional stiffness of the wing. For a dynamical system such as an oscillating wing, the presence of a variable torsional stiffness renders the system to be time variant, that is, the parameters defining the system vary with time. This time-varying stiffness is used herein to control airfoil oscillations. It is also advantageous to have a variable stiffness wing in the case of morphing[1] wings as this allows the wing to hold the load while still being flexible enough to allow morphing of the airfoil for different flight regimes. A brief description of the present approach and results is given here. Details are available in the conference paper, AIAA Paper 2002-5599.

Aeroelastic Applications of Neural Networks

Figure 1 shows the comparison between a doublet-panel code and the theoretical Wagner solution[3] for the unsteady lift over a symmetric airfoil with a variable-amplitude pitching motion. As can be seen from the comparison made in Fig. 1, the results from an unsteady doublet-panel code developed here and from theory are very similar. In the present work, this unsteady doublet-panel method is linked to a structural dynamics solver based on the method of matrix exponential time marching.[4] Analytical solutions to dynamic aeroelastic problems are few,[3] and the presence of arbitrarily time-varying coefficients in the structural dynamic equations prevents analytical solutions. When numerical methods are used to model the dynamic aeroelastic behavior, the coupling between the aerodynamics and structural behavior can lead to very large computational costs. Most applications of neural networks in aeroelasticity have been in system parameter estimation for aircraft control or in flutter control from experimental data.[5,6] A recent work in the application of neural networks to static nonlinear aeroelastic problems in morphing wings[7] describes the procedure of aeroelastic optimization of an adaptable bump on the surface of an airfoil.

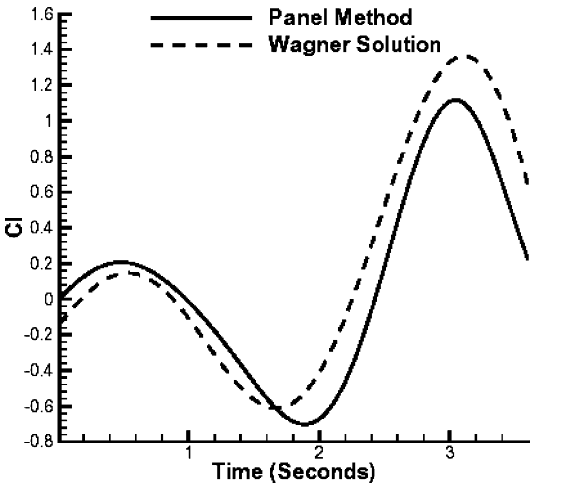

Fig. 1 Unsteady lift over a pitching airfoil with pitch angle $\alpha(t) = 0.1t\sin(2t)$ as computed by the doublet-panel code and compared with the Wagner solution.

The presence of varying coefficients in the differential equation of motion of a system significantly complicates the analysis even for a linear system. For example, traditional approaches to determine the stability, such as the Routh–Hurwitz criterion for continuous-time systems or Jury test[8] for discrete-time systems fail when the system is time varying. The second method of Liapunov, though applicable, is complicated because aeroelasticity involves nonconservative loading. Therefore herein, a stable time-domain solution for linear-time-variant systems is first explored and then utilized for training feed-forward neural networks such that a time-varying dynamic aeroelastic simulation is feasible.

Presence of noise is typical of recurrent neural networks and is one of the disadvantages of such neural networks. Hence in order to present a better match between the results as given by a recurrent neural network and those by a time-varying dynamical system, it is necessary to use smaller time steps However, use of smaller time steps increases the size of the dataset required for the network training. Large quantities of input data do not allow for accurate artificial-neural-network (ANN) training. Hence instead of using reduced time steps to make the neural-network representation of the time-varying system more accurate, it is more reasonable to study accurate numerical solvers for linear-time-varying systems that can use large time steps.

Recent advances in variable stiffness wings[2] can allow for exponentially saturating stiffness variation. Accordingly, the time-varying torsional stiffness is assumed as

$$k_\alpha(t) = k_0 + k_{01}\left(1 - c_1 e^{-c_2 t}\right) \tag{1}$$

where k_0 is the initial torsional stiffness and $k_0 + k_{01}$ is the maximum torsional stiffness. The rate of variation is controlled by the two parameters c_1 and c_2. Training of recurrent neural networks based on these numerical results for dynamic aeroelastic problems requires that the time step be small. Hence, we investigate a stable time-integration scheme for linear-time-varying systems using large time steps. The utilization of such a scheme for the training of feed-forward neural networks to represent the dynamic aeroelastic system is further investigated.

Response of Linear-Time-Varying Dynamical Systems

Linear-time-varying systems can be analyzed by the method of matrix exponential time marching.[4] Though an analytical solution

Received 26 July 2002; presented as Paper 2002-5599 at the AIAA/ISSMO 9th Symposium on Multidisciplinary Analysis and Optimization, Atlanta, GA, 4–6 September 2002; accepted for publication 7 April 2003. Copyright © 2004 by the authors. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0001-1452/04 $10.00 in correspondence with the CCC.

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